Numerical Differentiation

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Divided Difference

The variable x and function *y = f* (*x*) *in tabular form*

x	$f(x)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
$\dots$	$\dots$
x_{n-1}	$f(x_{n-1})$
x_n	$f(x_n)$

first divided difference

$$f(x_i, x_j) = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

second divided difference

$$f(x_i, x_j, x_k) = \frac{f(x_i, x_j) - f(x_j, x_k)}{x_i - x_k}$$

Tabular Representation of divided difference

x	$f(x)$			
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Form a difference table showing the values of given as *0*, *1*, *2*, *3*, *4*, *7*, and *9* , with $y=f(x)=x^3$

Function		Divided Differences			
x	$f(x) = x^3$	First	Second	Third	Fourth
0	0				
		1			
1	1		4		
		13		1	
3	27		8		0
		37		1	
4	64		14		0
		93		1	
7	343		20		
		193			
9	729				

```
double x[6]={0,1,3,4,7,9};
double f[6];
double first[5], second[4], third[3], fourth[2];
int i;
for(i=0;i<6;i++)f[i]=pow(x[i],3); // function
for(i=0;i<5;i++)first[i]=(f[i]-f[i+1])/(x[i]-x[i+1]);
for(i=0;i<4;i++)second[i]=(first[i]-first[i+1])/(x[i]-x[i+2]);
for(i=0;i<3;i++)third[i]=(second[i]-second[i+1])/(x[i]-x[i+3]);
for(i=0;i<2;i++)fourth[i]=(third[i]-third[i+1])/(x[i]-x[i+4]);
```

Form a difference table showing the values of given as *0*, *1*, *2*, *3*, *4*, *7*, and *9* , with $y=f(x)=x^3$

"C:\Documents and Settings\Pierre CHEN\My Documents\Debug\d_diff.exe"

```
0        1        27       64       343      729
1        13       37       93       193
4        8        14       20
1        1        1
0        0
Press any key to continue_
```

Difference

If the values of *x* in a table are equally spaced (a constant *h*), then $x_k = x_0 + k.h$

the first differences: $$\Delta f_k = f_{k+1} - f_k$$

second differences: $$\Delta^2 f_k = \Delta(\Delta f_k) = \Delta f_{k+1} - \Delta f_k$$

n^{th} differences:

$$\Delta^n f_k = \Delta(\Delta^{n-1} f_k) = \Delta^{n-1} f_{k+1} - \Delta^{n-1} f_k$$

Difference

Function			Differences				
k	x_k	f_k	Δf_k	$\Delta^2 f_k$	$\Delta^3 f_k$	$\Delta^4 f_k$	...
1	1	1					
			7				
2	2	8		12			
			19		6		
3	3	27		18		0	
			37		6		
4	4	64		24		0	
			61		6		
5	5	125		30		0	
			91		6		
6	6	216		36		0	
			127		6		
7	7	343		42			
			169				
8	8	512					

Ex1.

For

$$f(x) = x^4 - 5x^3 + 3x + 4$$

construct the difference table and x=0,1,2…,99

Draw the curve of each order of divided difference in Excel

Difference Using the binomial expansion

For k=0, n=1,2,3 and 4,

we have

$$\Delta f_0 = f_2 - f_1$$

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0$$

$$\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0$$

$$\Delta^4 f_0 = f_4 - 4f_3 + 6f_2 - 4f_1 + f_0$$

Or, more generally :

$$\Delta^n f_k = f_{k+n} - n f_{k+n-1} + \frac{n(n-1)}{2!} f_{k+n-2} + \ldots + (-1)^{n-1} n f_{k+1} + (-1)^n f_k$$

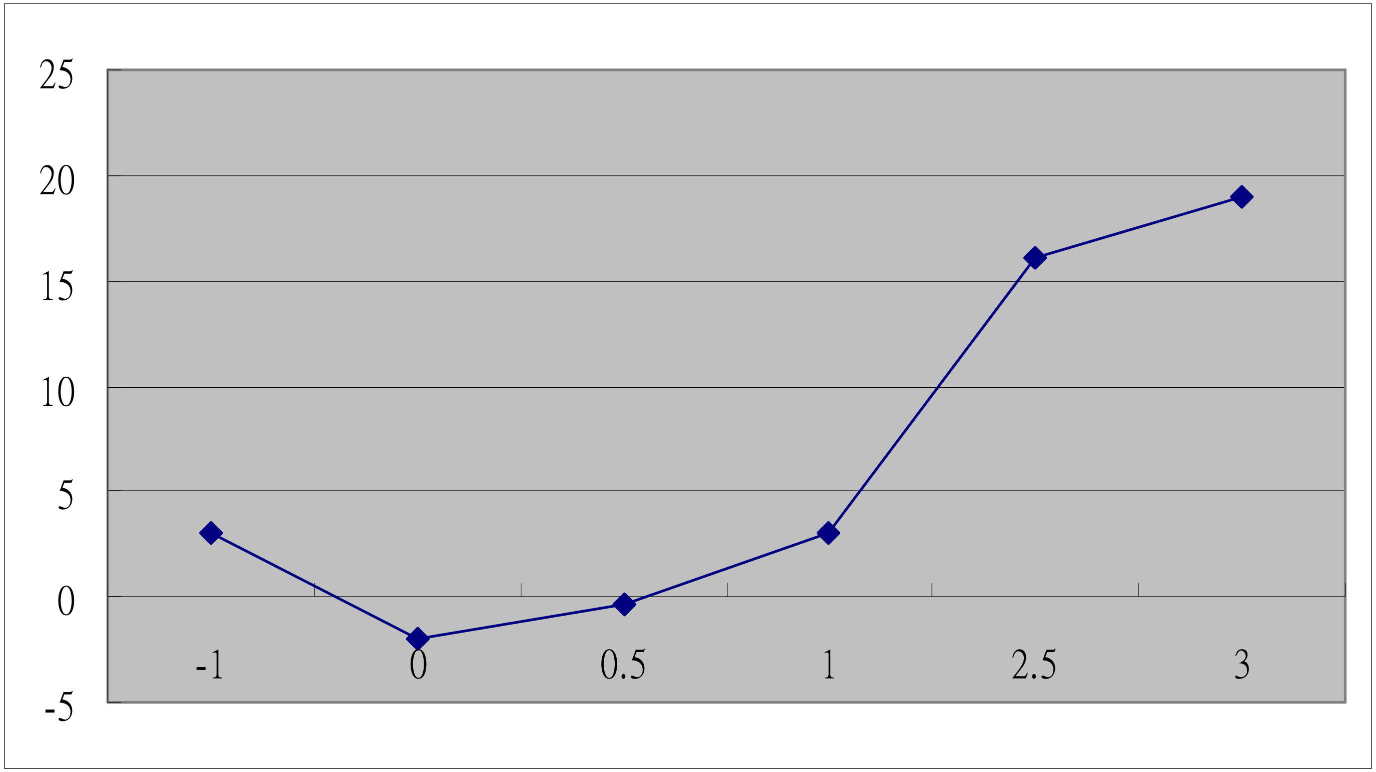
Newton's Divided Difference

While $x_0\, x_1\, x_2$, , , …, x_n are not equally spaced, we can use

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$$

Ex.2 Use Newton's divided-difference method to compute *f* (*2*) from

x	-1.0	0.0	0.5	1.0	2.5	3.0
$y = f(x)$	3.0	-2.0	-0.375	3.0	16.125	19.0



Ans: 12

Newton's Divided Difference Interpolation Method

x	3	1	5	6
$f(x)$	1	-3	2	4

$x_0 = 3$	$f[x_0] = 1$			
		$f[x_0, x_1] = 2$		
$x_1 = 1$	$f[x_1] = -3$		$f[x_0, x_1, x_2] = -\frac{3}{8}$	
		$f[x_1, x_2] = \frac{5}{4}$		$f[x_0, x_1, x_2, x_3] = \frac{7}{40}$
$x_2 = 5$	$f[x_2] = 2$		$f[x_1, x_2, x_3] = \frac{3}{20}$	
		$f[x_2, x_3] = 2$		
$x_3 = 6$	$f[x_3] = 4$			

$$
\begin{aligned}
p(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_0) \\
&\quad + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
&= 1 + 2(x - 3) - \frac{3}{8}(x - 3)(x - 1) + \frac{7}{40}(x - 3)(x - 1)(x - 5).
\end{aligned}
$$